

The extended Burnside ring and module categories

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Abstract

In this note an ‘extended Burnside ring’ is defined, generated by classes of semisimple module categories over $\text{Rep}(G)$ with quasifibre functors. Here G is a finite group and representations are taken over an algebraically closed field of characteristic 0. It is shown that this is equivalent to a ring generated by centrally extended G -sets and hence the name. Ring homomorphisms into the multiplicative group of the field are computed with an explicit formula and tables of these homomorphisms are given for the groups S_4 and S_5 which are of particular interest in the context of reductive algebraic groups.

Résumé

L’anneau de Burnside étendu et catégories de modules. Dans cette note un ‘Anneau de Burnside étendu’ est défini, généré par des classes de catégories de modules semisimples sur $\text{Rep}(G)$ avec des foncteurs quasifibres. Ici G est un groupe fini, et des représentations sont prises sur un corps algébriquement clos de caractéristique nulle. Il est démontré que ceci équivaut à un anneau généré par des G -ensembles centralement étendus, d’où le nom. Des homomorphismes d’anneau dans le groupe multiplicatif du corps sont calculées avec une formule explicite et des tableaux de ces homomorphismes sont fournis pour les groupes S_4 et S_5 qui sont d’un intérêt particulier dans le contexte de groupes algébriques réductifs.

Version française abrégée

L’anneau de Burnside étendu $\tilde{b}(G)$ de G est défini comme étant le groupe abélien généré par des classes d’équivalence de catégories de modules semisimples M sur $\text{Rep}(G)$ qui ont une fidèle $\text{Rep}(G)$ -foncteur de module $w : M \rightarrow \text{Rep}(G)$. Un tel foncteur est appelé un *foncteur quasifibre* [3]. Le groupe contient la relation $[M] + [N] = [M \oplus N]$ et une multiplication d’anneau peut être défini par $[M] \cdot [N] = [Fun(M, N)]$, où $Fun(M, N)$ dénote la catégorie de foncteurs de modules de M à N . Il peut être démontré que $Fun(M, N)$ est une catégorie de module en $\tilde{b}(G)$ en observant que la catégorie de module est équivalent à la catégorie de A -modules de droite en $\text{Rep}(G)$ pour quelques algèbres associatifs A en $\text{Rep}(G)$ [3, Prop. 4.1]. On peut également prouver que l’anneau est commutatif et associatif en considérant cette équivalence. Bezrukavnikov et Ostrik ont prouvé qu’une catégorie de module en $\tilde{b}(G)$ correspond à une extension centrale d’un G -ensemble fini [3, Theorem 3]. Ce sont des G -ensembles avec les données d’une extension centrale par k^* de chaque stabilisateur de point de telle manière que les extensions sont équivariantes sous l’action de G . Ces données peuvent être décrites par un groupoïde d’une façon ‘libre de coordonnées’. Ces groupoïdes qui correspondent à des éléments de $\tilde{b}(G)$ s’appellent des groupoïdes d’extension et peuvent être classées par paires (H, μ) où H est un sous-groupe de G représentant un G -ensemble fini et μ est un

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cocycle sur H qui spécifie une extension centrale de H . Le groupe cohomologue $H^2(H, k^*)$ sera noté $M(H)$. On multiplie des groupoïdes d'extension d'une façon qui correspond au produit de G -ensembles étendus par la somme des cocycles. Ceci correspond au produit défini dans l'anneau de Burnside étendu, et donc on peut considérer $\tilde{b}(G)$ comme étant l'anneau généré par les groupoïdes d'extension.

Dans la Section 2 on trouve une formule explicite pour les homomorphismes d'anneau de $\tilde{b}(G)$ à k^* . Cette formule est donnée par l'Equation (2) où ϕ est un caractère linéaire de $M(H)$. On démontre que ces formules définissent des homomorphismes d'anneau r distinct où r est le rang de l'anneau de Burnside comme groupe abélien libre, d'où,

Théorème 2.5. *Supposons que $\tilde{B}(G)$ soit le k -algèbre produit de $\tilde{b}(G)$ en tensoriant avec k . Supposons que Λ soit l'ensemble de toutes les homomorphismes d'anneau distinctes f_H^ϕ construites ci-dessus. Une fois tensorisé avec k , la somme des homomorphismes en Λ est un isomorphisme $\tilde{B}(G) \rightarrow \bigoplus_\Lambda k$. En plus, Λ est l'ensemble de toutes les homomorphismes d'anneau de $\tilde{b}(G)$ jusqu'à k .*

Les tableaux étendus de marques pour les groupes S_4 et S_5 sont imprimés dans les Tableaux 1 et 2 respectivement ainsi que les valeurs de la fonction m qui assigne à une catégorie de module le nombre d'objets simples qu'elle contient. Une application de cette théorie, $m(\text{Fun}(M, N))$ peut être facilement calculée de $m(M)$ et $m(N)$ en utilisant les données des tableaux.

Les groupes S_4 et S_5 sont essentielles dans la théorie de représentation puisqu'elles sont deux des types de groupes d'éléments des groupes réductifs algébriques. L'importance de la fonction m sur ces groupes d'éléments est prise en considération dans un travail récent de Bezrukavnikov, Finkelberg et Ostrik.

1 Introduction

Let G be a finite group, k an algebraically closed field of characteristic 0 and $\text{Rep}(G)$ the category of representations of G over k .

We define the *extended Burnside ring* $\tilde{b}(G)$ of G to be the abelian group generated by equivalence classes of semisimple module categories \mathcal{M} over $\text{Rep}(G)$ that have a faithful exact module functor $\omega : \mathcal{M} \rightarrow \text{Rep}(G)$. Such a functor will be called a *quasifibre functor*. This group carries the relation $[M] + [N] = [M \oplus N]$ and ring multiplication is defined by $[M] \cdot [N] = [\text{Fun}(M, N)]$, where $\text{Fun}(M, N)$ denotes the category of module functors from M to N . We now give a brief proof that $\text{Fun}(M, N)$ is indeed a semisimple module category with quasifibre functor, making the multiplication above well defined. A module category in $\tilde{b}(G)$ is equivalent to the category of right A modules in $\text{Rep}(G)$ for some associative algebra A in $\text{Rep}(G)$ [3, Prop. 4.1]. Thus $\text{Fun}(M, N)$ is equivalent to $\text{Fun}(\text{Mod}_{\text{Rep}(G)}(A), \text{Mod}_{\text{Rep}(G)}(B))$ for some algebras A, B , which is in turn equivalent to the category of $A - B$ bimodules in $\text{Rep}(G)$. This category is equivalent to $\text{Mod}_{\text{Rep}(G)}(B \otimes A^{\text{op}})$. This is a semisimple category (as $\text{char}(k) = 0$, [8, p. 3]) and the forgetful functor is a quasifibre functor. A semisimple ring is isomorphic to its opposite and so it follows that multiplication is associative and commutative.

Bezrukavnikov and Ostrik have proved that a module category in $\tilde{b}(G)$ corresponds to a centrally extended finite G -set [3, Theorem 3]. These are G -sets with the data of a central extension by k^* of each point stabiliser such that the extensions are equivariant under the action of G . This data can be described by a groupoid in a 'coordinate free' manner. Any G -set X defines a groupoid \mathcal{G}_X over the base X called the action groupoid, where arrows correspond to the action of G . A centrally extended finite G -set corresponds to a central extension groupoid of \mathcal{G}_X (see [7, p.12]). The transitive centrally extended G -sets are classified by a subgroup H of G (corresponding to a transitive G -sets) and one cocycle $\mu \in H^2(H, k^*) =: M(H)$ (to specify a central extension of H). We shall call centrally extended groupoids defined by a finite G -set *extended groupoids*.

The natural way to multiply extended G -sets is to extend the diagonal action of G on the product set in a way which corresponds to addition of cocycles. In the language of groupoids we define this product by taking the fibre product of two extended groupoids over the group G and quotienting by the kernel of the multiplication map on k^* . The disjoint union of extended groupoids corresponds to addition of the corresponding module categories.

For M a module category in $\tilde{b}(G)$ with corresponding groupoid $\tilde{\mathcal{G}}$, $\text{Fun}(M, M)$ corresponds to the product $\tilde{\mathcal{G}} \cdot \tilde{\mathcal{G}}$ [3, §5.1]. Both multiplications are commutative, so it follows that the product of any two extended groupoids corresponds to the product in $\tilde{b}(G)$. Thus we have reformulated the ring $\tilde{b}(G)$ as a ring of extended groupoids on G with addition and multiplication described above.

2 Main Results

In this section we find all homomorphisms from the extended Burnside ring $\tilde{b}(G)$ to the multiplicative group of the field k^* . The class of the G -set G/H will be denoted $\langle H \rangle$ and the class $[G/H]$ extended by cocycle μ is denoted $\langle H^\mu \rangle$.

If $\langle H \rangle \cdot \langle K \rangle$ denotes multiplication in the Burnside ring then $\langle H \rangle \cdot \langle K \rangle = \sum_{KgH \in K \backslash G/H} \langle H \cap K^g \rangle$ [5, §80B].

In the extended Burnside ring multiplication of extended G -sets occurs by first multiplying the underlying G -sets as in the Burnside ring and then extending by the cocycles in a natural way that relates to adding the cocycles as follows:

$$\langle H^\mu \rangle \cdot \langle K^\nu \rangle = \sum_{KgH \in K \backslash G/H} \langle (H \cap K^g)^{\text{res}_{H, H \cap K^g}(\mu) + \text{res}_{K^g, H \cap K^g}(\nu^g)} \rangle, \quad (1)$$

where $\text{res}_{K, H}$ denotes the restriction of cocycles from a group K to a subgroup H and the cocycle ν^g on K^g is defined by $\nu^g(x^g, y^g) = \mu(x, y)$.

Proposition 2.1. *Let $H \leq G$ and ϕ a linear character of the abelian group $M(H)$. Define a k^* valued function f_H^ϕ on \tilde{b} by:*

$$f_H^\phi(\langle K^\mu \rangle) = \frac{1}{|K|} \sum_{g \in \mathcal{S}} \phi(\text{res}_{K^g, H}(\mu^g)), \quad (2)$$

where $\mathcal{S} = \{g \in G \mid H \subseteq K^g\}$, $K \leq G$ and $\mu \in M(K)$. Then f_H^ϕ is a ring homomorphism.

Proof. Suppose K and N are subgroups of G and μ and ν are respective cocycles on K and N . For $x \in G$ let $n_x = |K \cap N^x|$ and $\mathcal{S}_x = \{g \in G \mid H \subseteq K^g \cap N^{xg}\}$. Applying f_H^ϕ to 1 we have:

$$f_H^\phi(\langle K^\mu \rangle \cdot \langle N^\nu \rangle) = \sum_{NxK} \frac{1}{n_x} \sum_{g \in \mathcal{S}_x} \phi(\text{res}_{K^g, H}(\mu^g) + \text{res}_{N^{xg}, H}(\nu^{xg})). \quad (3)$$

□

Clearly the size of a double coset NxK is equal to the set $x^{-1}NxK$ since left multiplication by a group element defines a bijective correspondence. The set $x^{-1}NxK$ is actually the product of the subgroups N^x and K . Thus, $|K \cap N^x| = \frac{|K||N^x|}{|x^{-1}NxK|} = \frac{|K||N|}{|NxK|}$. If $y \in NxK$, then $L \subseteq K^g \cap N^{xg}$ implies that for some $k \in K$ and $h \in N$, $L \subseteq K^g \cap N^{hykg} = K^{kg} \cap N^{ykg}$. A result in Cartan and Eilenberg [4, Ch. 12] implies that $\text{res}_{K^g, H}(\mu^g) = \text{res}_{K^{kg}, H}(\mu^{kg})$.

Hence from Equation (3) we derive:

$$\begin{aligned}
f_H^\phi(\langle K^\mu \rangle \cdot \langle N^\nu \rangle) &= \frac{1}{|K||N|} \sum_{x \in G} \sum_{H \subseteq K^g \cap N^{xg}} \phi(\text{res}_{K^g, H}(\mu^g) + \text{res}_{N^{xg}, H}(\nu^{xg})) \\
&= \frac{1}{|K||N|} \sum_{H \subseteq K^g} \sum_{H \subseteq N^k} \phi(\text{res}_{K^g, H}(\mu^g)) \phi(\text{res}_{N^k, H}(\nu^k)) \\
&= f_H^\phi(\langle K^\mu \rangle) f_H^\phi(\langle N^\nu \rangle).
\end{aligned}$$

We now show that the number of distinct homomorphisms is equal to the rank of $\tilde{b}(G)$ as a free abelian group.

Let $H, K \leq G$ be nonconjugate subgroups and without loss of generality assume that $H \not\subseteq K$. Let ϕ be a linear character of $M(H)$ and χ be a linear character of $M(K)$. Then by Proposition 2.1 $f_H^\phi(\langle K \rangle) = 0$. But, if 1 denotes the trivial cocycle on K , $f_K^\chi(\langle K \rangle) = \sum_{g \in N_G(K)} \chi(1) \neq 0$. Hence $f_H^\phi \neq f_K^\chi$.

Let H be a subgroup of G and let N denote the normaliser of H in G . We define an action of N on the group of linear characters of $M(H)$ via $\phi^g(\mu) := \phi(\mu^g)$ for ϕ a linear character of $M(H)$ and $g \in N$.

Lemma 2.2. *Let ϕ and ψ be linear characters of $M(H)$. If the ring homomorphisms f_H^ϕ and f_H^ψ are equal, then there exists $g \in N$ such that $\phi = \psi^g$.*

Proof. Let $\mu \in M(H)$. Applying f_H^ϕ and f_H^ψ to $\langle H^\mu \rangle$ we get $\sum_{g \in N} \phi(\mu^g) = \sum_{g \in N} \psi(\mu^g)$, and so

$$\sum_{g \in N} \phi^g(\mu) = \sum_{g \in N} \psi^g(\mu) \quad \forall \mu \in M(H). \quad (4)$$

The linear characters of $M(H)$ form a linearly independent set in the character ring and hence Equation (4) implies that $\phi = \psi^g$ for some $g \in N$. \square

Hence we have proved the theorem,

Theorem 2.3. *The homomorphisms f_H^ϕ described above define a set of r distinct ring homomorphisms from $\tilde{b}(G)$ to the complex numbers as H ranges over conjugacy class representatives of subgroups of G and ϕ over a set of orbit representatives of linear characters of $M(H)$ under the action of N . Moreover, r is equal to the rank of $\tilde{b}(G)$ as an additive free abelian group.*

The following technical lemma is proved in Benson [1, p. 163]:

Lemma 2.4. *If R is a commutative ring and S an integral domain then any set of distinct ring homomorphisms $f_i : R \rightarrow S$ is linearly independent over S .*

This leads to the proof of the final theorem,

Theorem 2.5. *Let $\tilde{B}(G)$ be the k -algebra produced from $\tilde{b}(G)$ by tensoring with k . Let Λ be the set of all the distinct ring homomorphisms f_H^ϕ constructed above. After tensoring with k the sum of the homomorphisms in Λ is an isomorphism: $\sum_{\Lambda} f_H^\phi : \tilde{B}(G) \rightarrow \bigoplus_{\Lambda} k$. Moreover Λ is the set of all the ring homomorphisms from $\tilde{b}(G)$ to k .*

Proof. Lemma 2.4 implies we have a set of $|\Lambda|$ independent homomorphisms and so the map $\sum_{\Lambda} f_H^\phi$ must be surjective. The dimension of $\tilde{B}(G)$ as a vector space over k is equal to $|\Lambda|$ and so $\sum_{\Lambda} f_H^\phi$ is an isomorphism. If we had another ring homomorphism from $\tilde{b}(G)$ to k distinct from all those in Λ then as above we can construct a surjective k -linear map from $\tilde{B}(G)$ to a vector space with strictly larger dimension. This is a contradiction, hence Λ is a complete set of homomorphisms. \square

3 Application

We present the extended table of marks for the symmetric groups S_4 and S_5 in Tables 1 and 2. In these tables D_8 denotes the dihedral group of order 8, $V_1 = \langle (12), (34) \rangle$ and $V_2 = \langle (12)(34), (13)(24) \rangle$. The subgroups of S_5 of order 10 and 20 (unique up to conjugation) are denoted H_{10} and H_{20} respectively and H_6 is used for the nonabelian subgroup of order 6 not isomorphic to S_3 . All Schur multipliers have order at most 2, so we use H' for the nontrivial extension. The columns of the tables correspond to values of the ring homomorphisms of $\tilde{b}(S_4)$ ordered as for the rows. Appended to the tables are the values of the function $m : \tilde{b}(G) \rightarrow \mathbb{Z}$ taking a module category to the number of simple objects it contains.

The tables were computed by lifting data from the ordinary table of marks and the following lemma.

Lemma 3.1. *Let $H \leq K \leq G$ and suppose $|M(K)| = |M(H)| = 2$ and 2 does not divide the index $|K : H|$. Then $f'_H(\langle K' \rangle) = -f_H(\langle K \rangle)$.*

Proof. Let $\mu \in M(K)$ and $\nu \in M(H)$ be non-trivial cocycles. There exists a corestriction function $\text{cor}_{H,K} : M(H) \rightarrow M(K)$ with the property that $\text{res}_{K,H}(\text{cor}_{H,K}(\nu)) = |K : H|\nu$ [6, Ch. 1]. Thus, $\text{res}_{K,H}(\text{cor}_{H,K}(\nu)) = \nu$ and so ν must corestrict to the nontrivial cocycle μ and $\text{res}_{K,H}(\mu) \neq 1$. The lemma now follows from Equation (2). \square

An application of these data tables is the ability to compute the number of simple objects in the category $\text{Fun}(M, N)$ where M and N are semisimple module categories over $\text{Rep}(G)$ with quasifibre functors. An explicit example occurs in the paper of Bezrukavnikov, Finkelberg and Ostrik [2]. In the proof of Theorem 3 of *loc. cit.* it is required to show that for $M, N \in \tilde{b}(S_4)$ such that $m(M \cdot M) = m(N \cdot N) = 5$ and $m(M \cdot N) = 3$, either $M \cdot M = S_4$ or $N \cdot N = S_4$. This follows immediately from Table 1 since $m(M \cdot M) = 5$ implies $M \in \{S_3, S_4, S'_4\}$ and we know $M \neq N$.

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																		m
1	24																	1
$\langle (12)(34) \rangle$	12	4																2
$\langle (12) \rangle$	12	0	2															2
$\langle (123) \rangle$	8	0	0	2														3
$\langle (1234) \rangle$	6	2	0	0	2													4
S_3	4	0	2	1	0	1												3
V_1	6	2	2	0	0	0	2											4
V_2	6	6	0	0	0	0	0	6										4
D_8	3	3	1	0	1	0	1	3	1									5
A_4	2	2	0	2	0	0	0	2	0	2								4
S_4	1	1	1	1	1	1	1	1	1	1	1							5
V'_1	6	2	2	0	0	0	2					-2						1
V'_2	6	6	0	0	0	0	0	6				0	-6					1
D'_8	3	3	1	0	1	0	1	3	1			1	3	-1				2
A'_4	2	2	0	2	0	0	0	2	0	2		0	-2	0	-2			3
S'_4	1	1	1	1	1	1	1	1	1	1	1	1	1	-1	1	-1		3

Table 1: The extended table of marks of S_4 . Le tableau étendu de marques pour le groupe S_4 .

Table 2: The extended table of marks of S_5 . Le tableau étendu de marques pour le groupe S_5 .

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